

**NCTM
Standards
(2000)
for Grades
Pre-K
through 12th**

NCTM Standards (2000) for Grades Pre-K through 2nd

Number and Operations Standard for Grades Pre-K-2

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In prekindergarten through grade 2 all students should—
Understand numbers, ways of representing numbers, relationships among numbers, and number systems	<ul style="list-style-type: none">• count with understanding and recognize "how many" in sets of objects;• use multiple models to develop initial understandings of place value and the base-ten number system;• develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections;• develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers;• connect number words and numerals to the quantities they represent, using various physical models and representations;• understand and represent commonly used fractions, such as $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$.
Understand meanings of operations and how they relate to one another	<ul style="list-style-type: none">• understand various meanings of addition and subtraction of whole numbers and the relationship between the two operations;• understand the effects of adding and subtracting whole numbers;• understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally.
Compute fluently and make reasonable estimates	<ul style="list-style-type: none">• develop and use strategies for whole-number computations, with a focus on addition and subtraction;• develop fluency with basic number combinations for addition and subtraction;• use a variety of methods and tools to compute, including objects, mental computation, estimation, paper and pencil, and calculators.

Algebra Standard for Grades Pre-K-2

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In prekindergarten through grade 2 all students should—
Understand patterns, relations, and functions	<ul style="list-style-type: none">• sort, classify, and order objects by size, number, and other properties;• recognize, describe, and extend patterns such as sequences of sounds and shapes or simple numeric patterns and translate from one representation to another;• analyze how both repeating and growing patterns are generated.
Represent and analyze mathematical situations and structures using algebraic symbols	<ul style="list-style-type: none">• illustrate general principles and properties of operations, such as commutativity, using specific numbers;• use concrete, pictorial, and verbal representations to develop an understanding of invented and conventional symbolic notations.
Use mathematical models to represent and understand quantitative relationships	<ul style="list-style-type: none">• model situations that involve the addition and subtraction of whole numbers, using objects, pictures, and symbols.
Analyze change in various contexts	<ul style="list-style-type: none">• describe qualitative change, such as a student's growing taller;• describe quantitative change, such as a student's growing two inches in one year.

Geometry Standard for Grades Pre-K-2

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In prekindergarten through grade 2 all students should—
Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships	<ul style="list-style-type: none"> • recognize, name, build, draw, compare, and sort two- and three-dimensional shapes; • describe attributes and parts of two- and three-dimensional shapes; • investigate and predict the results of putting together and taking apart two- and three-dimensional shapes.
Specify locations and describe spatial relationships using coordinate geometry and other representational systems	<ul style="list-style-type: none"> • describe, name, and interpret relative positions in space and apply ideas about relative position; • describe, name, and interpret direction and distance in navigating space and apply ideas about direction and distance; • find and name locations with simple relationships such as "near to" and in coordinate systems such as maps.
Apply transformations and use symmetry to analyze mathematical situations	<ul style="list-style-type: none"> • recognize and apply slides, flips, and turns; • recognize and create shapes that have symmetry.
Use visualization, spatial reasoning, and geometric modeling to solve problems	<ul style="list-style-type: none"> • create mental images of geometric shapes using spatial memory and spatial visualization; • recognize and represent shapes from different perspectives; • relate ideas in geometry to ideas in number and measurement; • recognize geometric shapes and structures in the environment and specify their location.

Measurement Standard for Grades Pre-K-2

Expectations

<p>Instructional programs from prekindergarten through grade 12 should enable all students to—</p>	<p>In prekindergarten through grade 2 all students should—</p>
<p>Understand measurable attributes of objects and the units, systems, and processes of measurement</p>	<ul style="list-style-type: none"> • recognize the attributes of length, volume, weight, area, and time; • compare and order objects according to these attributes; • understand how to measure using nonstandard and standard units; • select an appropriate unit and tool for the attribute being measured.
<p>Apply appropriate techniques, tools, and formulas to determine measurements</p>	<ul style="list-style-type: none"> • measure with multiple copies of units of the same size, such as paper clips laid end to end; • use repetition of a single unit to measure something larger than the unit, for instance, measuring the length of a room with a single meterstick; • use tools to measure; • develop common referents for measures to make comparisons and estimates.

Data Analysis and Probability Standard for Grades Pre-K-2

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In prekindergarten through grade 2 all students should—
Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them	<ul style="list-style-type: none"> • pose questions and gather data about themselves and their surroundings; • sort and classify objects according to their attributes and organize data about the objects; • represent data using concrete objects, pictures, and graphs.
Select and use appropriate statistical methods to analyze data	<ul style="list-style-type: none"> • describe parts of the data and the set of data as a whole to determine what the data show.
Develop and evaluate inferences and predictions that are based on data	<ul style="list-style-type: none"> • discuss events related to students' experiences as likely or unlikely.
Understand and apply basic concepts of probability	

Problem Solving Standard for Grades Pre-K-2

Instructional programs from prekindergarten through grade 12 should enable all students to— <ul style="list-style-type: none"> • build new mathematical knowledge through problem solving; • solve problems that arise in mathematics and in other contexts; • apply and adapt a variety of appropriate strategies to solve problems; • monitor and reflect on the process of mathematical problem solving.
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Reasoning and Proof Standard for Grades Pre-K-2

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

Communication Standard for Grades Pre-K-2

Instructional programs from prekindergarten through grade 12 should enable all students to—

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely.

Connections Standard for Grades Pre-K-2

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics.

Representation Standard for Grades Pre-K-2

Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Overview

Standards for Grades Pre-K-2

Developing a solid mathematical foundation from prekindergarten through second grade is essential for every child. In these grades, students are building beliefs about what mathematics is, about what it means to know and do mathematics, and about themselves as mathematics learners. These beliefs influence their thinking about, performance in, and attitudes toward, mathematics and decisions related to studying mathematics in later years.



Children develop many mathematical concepts, at least in their intuitive beginnings, even before they reach school age. Infants spontaneously recognize and discriminate among small numbers of objects, and many preschool children possess a substantial body of informal mathematical knowledge. Adults can foster children's mathematical development from the youngest ages by providing environments rich in language and where thinking is encouraged, uniqueness is valued, and exploration is supported.

Children are likely to enter formal school settings with different levels of mathematics understanding, reflecting their opportunity to have learned mathematics. Some children will need additional support so that they do not start school at a disadvantage. Early assessments should be used not to sort children but to gain information for teaching and for potential early interventions.

All students deserve high-quality programs that include significant mathematics presented in a manner that respects both the mathematics and the nature of young children. These programs must build on and extend students' intuitive and informal mathematical knowledge. They must be grounded in a knowledge of child development and provide environments that encourage students to be active learners and accept new challenges. They need to develop a strong conceptual framework while encouraging and developing students' skills and their natural inclination to solve problems.

At the core of mathematics programs in prekindergarten through grade 2 are the Number and Operations and Geometry Standards. For example, it is absolutely essential that students develop a solid understanding of the base-ten numeration system in prekindergarten through grade 2. They must recognize that the word ten may represent a single entity (1 ten) or ten separate units (10 ones) and that these representations are interchangeable. Using concrete materials and calculators in appropriate ways can help students learn these concepts.

Understandings of patterns, measurement, and data contribute to the understanding of number and geometry and are learned in conjunction with them. Similarly, the Process Standards of Problem Solving, Reasoning and Proof, Communication, Connections, and Representation both

support and augment the Content Standards. Even at this age, guided work with calculators can enable students to explore number and patterns, focus on problem-solving processes, and investigate realistic applications. See, for example, the problem in figure 1.

Makes a New Number

	Use a calculator. Start with 78. Do not press clear. Make the next number:	What numbers did you add or subtract?
	98	
Next number →	48	
	10	
	118	
	119	

Fig. 1. A calculator activity to help develop understanding of place value

In the elementary grades, it often happens that specific blocks of time are not allotted to instruction in particular subjects. It is essential for students in the elementary grades to study mathematics for an hour a day under the guidance of teachers who enjoy mathematics and are prepared to teach it well. This basic requirement takes thoughtful arrangements of scheduling and staffing--whether by shared teaching responsibilities, the use of mathematics specialists, or other creative administrative means.

NCTM Standards (2000) for Grades 3rd through 5th

Number and Operations Standard for Grades 3-5

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 3–5 all students should—
Understand numbers, ways of representing numbers, relationships among numbers, and number systems	<ul style="list-style-type: none">• understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals;• recognize equivalent representations for the same number and generate them by decomposing and composing numbers;• develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers;• use models, benchmarks, and equivalent forms to judge the size of fractions;• recognize and generate equivalent forms of commonly used fractions, decimals, and percents;• explore numbers less than 0 by extending the number line and through familiar applications;• describe classes of numbers according to characteristics such as the nature of their factors.
Understand meanings of operations and how they relate to one another	<ul style="list-style-type: none">• understand various meanings of multiplication and division;• understand the effects of multiplying and dividing whole numbers;• identify and use relationships between operations, such as division as the inverse of multiplication, to solve problems;• understand and use properties of operations, such as the distributivity of multiplication over addition.
Compute fluently and make reasonable estimates	<ul style="list-style-type: none">• develop fluency with basic number combinations for multiplication and division and use these combinations to mentally compute related problems, such as 30×50 ;• develop fluency in adding, subtracting, multiplying, and dividing whole numbers;• develop and use strategies to estimate the results of whole-number computations and to judge the reasonableness of such results;

	<ul style="list-style-type: none"> • develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experience; • use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions and decimals; • select appropriate methods and tools for computing with whole numbers from among mental computation, estimation, calculators, and paper and pencil according to the context and nature of the computation and use the selected method or tools
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Algebra Standard for Grades 3-5

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 3–5 all students should—
Understand patterns, relations, and functions	<ul style="list-style-type: none"> • describe, extend, and make generalizations about geometric and numeric patterns; • represent and analyze patterns and functions, using words, tables, and graphs.
Represent and analyze mathematical situations and structures using algebraic symbols	<ul style="list-style-type: none"> • identify such properties as commutativity, associativity, and distributivity and use them to compute with whole numbers; • represent the idea of a variable as an unknown quantity using a letter or a symbol; • express mathematical relationships using equations.
Use mathematical models to represent and understand quantitative relationships	<ul style="list-style-type: none"> • model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions.
Analyze change in various contexts	<ul style="list-style-type: none"> • investigate how a change in one variable relates to a change in a second variable; • identify and describe situations with constant or varying rates of change and compare them.

Geometry Standard for Grades 3-5

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 3–5 all students should—
Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships	<ul style="list-style-type: none"> • identify, compare, and analyze attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes; • classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids; • investigate, describe, and reason about the results of subdividing, combining, and transforming shapes; • explore congruence and similarity; • make and test conjectures about geometric properties and relationships and develop logical arguments to justify conclusions.
Specify locations and describe spatial relationships using coordinate geometry and other representational systems	<ul style="list-style-type: none"> • describe location and movement using common language and geometric vocabulary; • make and use coordinate systems to specify locations and to describe paths; • find the distance between points along horizontal and vertical lines of a coordinate system.
Apply transformations and use symmetry to analyze mathematical situations	<ul style="list-style-type: none"> • predict and describe the results of sliding, flipping, and turning two-dimensional shapes; • describe a motion or a series of motions that will show that two shapes are congruent; • identify and describe line and rotational symmetry in two- and three-dimensional shapes and designs.
Use visualization, spatial reasoning, and geometric modeling to solve problems	<ul style="list-style-type: none"> • build and draw geometric objects; • create and describe mental images of objects, patterns, and paths; • identify and build a three-dimensional object from two-dimensional representations of that object; • identify and draw a two-dimensional representation of a three-dimensional object; • use geometric models to solve problems in other areas of mathematics, such as number and measurement; • recognize geometric ideas and relationships and apply them to other disciplines and to problems that arise in the classroom or in everyday life.

Measurement Standard for Grades 3-5

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 3–5 all students should—
Understand measurable attributes of objects and the units, systems, and processes of measurement	<ul style="list-style-type: none">• understand such attributes as length, area, weight, volume, and size of angle and select the appropriate type of unit for measuring each attribute;• understand the need for measuring with standard units and become familiar with standard units in the customary and metric systems;• carry out simple unit conversions, such as from centimeters to meters, within a system of measurement;• understand that measurements are approximations and how differences in units affect precision;• explore what happens to measurements of a two-dimensional shape such as its perimeter and area when the shape is changed in some way.
Apply appropriate techniques, tools, and formulas to determine measurements	<ul style="list-style-type: none">• develop strategies for estimating the perimeters, areas, and volumes of irregular shapes;• select and apply appropriate standard units and tools to measure length, area, volume, weight, time, temperature, and the size of angles;• select and use benchmarks to estimate measurements;• develop, understand, and use formulas to find the area of rectangles and related triangles and parallelograms;• develop strategies to determine the surface areas and volumes of rectangular solids.

Data Analysis and Probability Standard for Grades 3-5

Expectations

<p>Instructional programs from prekindergarten through grade 12 should enable all students to—</p>	<p>In grades 3–5 all students should—</p>
<p>Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them</p>	<ul style="list-style-type: none"> • design investigations to address a question and consider how data-collection methods affect the nature of the data set; • collect data using observations, surveys, and experiments; • represent data using tables and graphs such as line plots, bar graphs, and line graphs; • recognize the differences in representing categorical and numerical data.
<p>Select and use appropriate statistical methods to analyze data</p>	<ul style="list-style-type: none"> • describe the shape and important features of a set of data and compare related data sets, with an emphasis on how the data are distributed; • use measures of center, focusing on the median, and understand what each does and does not indicate about the data set; • compare different representations of the same data and evaluate how well each representation shows important aspects of the data.
<p>Develop and evaluate inferences and predictions that are based on data</p>	<ul style="list-style-type: none"> • propose and justify conclusions and predictions that are based on data and design studies to further investigate the conclusions or predictions.
<p>Understand and apply basic concepts of probability</p>	<ul style="list-style-type: none"> • describe events as likely or unlikely and discuss the degree of likelihood using such words as <i>certain</i>, <i>equally likely</i>, and <i>impossible</i>; • predict the probability of outcomes of simple experiments and test the predictions; • understand that the measure of the likelihood of an event can be represented by a number from 0 to 1.

Problem Solving Standard for Grades 3-5

Instructional programs from prekindergarten through grade 12 should enable all students to—

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard for Grades 3-5

Instructional programs from prekindergarten through grade 12 should enable all students to—

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving.

Communication Standard for Grades 3-5

Instructional programs from prekindergarten through grade 12 should enable all students to—

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely.

Connections Standard for Grades 3-5

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics.

Representation Standard for Grades 3-5

Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Overview

Standards for Grades 3-5

Students enter grade 3 with an interest in learning mathematics. Nearly three-quarters of U.S. fourth graders report liking mathematics, seeing it as practical and important. If mathematics continues to be seen as interesting and understandable, students will remain engaged. If learning becomes simply a process of mimicking and memorizing, students' interest is likely to diminish.

Interwoven through the Content Standards for grades 3–5 are three crucial mathematical themes--multiplicative thinking, equivalence, and computational fluency. The focus on multiplicative reasoning develops knowledge that students build on as they move into the middle grades, where the emphasis is on proportional reasoning. As a part of multiplicative reasoning, students in grades 3–5 should build their understanding of fractions as a part of a whole and as division.

The concept of equivalence helps students learn different mathematical representations and offers a way to explore algebraic ideas. Students should develop computational fluency--efficient and accurate methods for computing that are based on well-understood properties and number relationships. For example, 298×42 can be thought of as $(300 \times 42) - (2 \times 42)$, or 41×16 can be computed by multiplying 41×8 to get 328 and then doubling 328 to get 656. When these three themes are emphasized, the expectations for grades 3–5 reinforce two major objectives of mathematics learning: making sense of mathematical ideas and acquiring the skills and understandings needed to solve problems.

In grades 3–5, algebraic ideas emerge and are investigated by children. For example, students in these grades are able to make a general statement about how one variable is related to another variable. If a sandwich costs \$3, you can figure out how many dollars any number of sandwiches cost by multiplying that number by 3. In this case, students have developed a model of a proportional relationship: the value of one variable is always 3 times the value of the other, or $C = 3 \times n$.

Given their central role in shaping the mathematics learning of students in these grades, teachers must recognize the need to develop mathematical expertise. Some elementary schools identify a "mathematics teacher-leader," who can support other teachers in their instruction and professional development. Other schools use "mathematics specialists" at the upper elementary grade levels, who assume primary responsibility for teaching mathematics to larger groups of students. Each of these models needs to be explored to enhance the mathematics education of students in grades 3–5

NCTM Standards (2000) for Grades 6th through 8th

Number and Operations Standard for Grades 6-8

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 6–8 all students should—
Understand numbers, ways of representing numbers, relationships among numbers, and number systems	<ul style="list-style-type: none">• work flexibly with fractions, decimals, and percents to solve problems;• compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line;• develop meaning for percents greater than 100 and less than 1;• understand and use ratios and proportions to represent quantitative relationships;• develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation;• use factors, multiples, prime factorization, and relatively prime numbers to solve problems;• develop meaning for integers and represent and compare quantities with them.
Understand meanings of operations and how they relate to one another	<ul style="list-style-type: none">• understand the meaning and effects of arithmetic operations with fractions, decimals, and integers;• use the associative and commutative properties of addition and multiplication and the distributive property of multiplication over addition to simplify computations with integers, fractions, and decimals;• understand and use the inverse relationships of addition and subtraction, multiplication and division, and squaring and finding square roots to simplify computations and solve problems.
Compute fluently and make reasonable estimates	<ul style="list-style-type: none">• select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply the selected methods;• develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use;• develop and use strategies to estimate the results of rational-number computations and judge the reasonableness of the results;• develop, analyze, and explain methods for solving problems involving proportions, such as scaling and finding equivalent ratios.

Algebra Standard for Grades 6-8

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 6–8 all students should—
Understand patterns, relations, and functions	<ul style="list-style-type: none"> • represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules; • relate and compare different forms of representation for a relationship; • identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations.
Represent and analyze mathematical situations and structures using algebraic symbols	<ul style="list-style-type: none"> • develop an initial conceptual understanding of different uses of variables; • explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope; • use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships; • recognize and generate equivalent forms for simple algebraic expressions and solve linear equations
Use mathematical models to represent and understand quantitative relationships	<ul style="list-style-type: none"> • model and solve contextualized problems using various representations, such as graphs, tables, and equations.
Analyze change in various contexts	<ul style="list-style-type: none"> • use graphs to analyze the nature of changes in quantities in linear relationships.

Geometry Standard for Grades 6-8

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 6–8 all students should—
Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships	<ul style="list-style-type: none"> • precisely describe, classify, and understand relationships among types of two- and three-dimensional objects using their defining properties; • understand relationships among the angles, side lengths, perimeters, areas, and volumes of similar objects; • create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity, and the Pythagorean relationship.
Specify locations and describe spatial relationships using coordinate geometry and other representational systems	<ul style="list-style-type: none"> • use coordinate geometry to represent and examine the properties of geometric shapes; • use coordinate geometry to examine special geometric shapes, such as regular polygons or those with pairs of parallel or perpendicular sides.
Apply transformations and use symmetry to analyze mathematical situations	<ul style="list-style-type: none"> • describe sizes, positions, and orientations of shapes under informal transformations such as flips, turns, slides, and scaling; • examine the congruence, similarity, and line or rotational symmetry of objects using transformations.
Use visualization, spatial reasoning, and geometric modeling to solve problems	<ul style="list-style-type: none"> • draw geometric objects with specified properties, such as side lengths or angle measures; • use two-dimensional representations of three-dimensional objects to visualize and solve problems such as those involving surface area and volume; • use visual tools such as networks to represent and solve problems; • use geometric models to represent and explain numerical and algebraic relationships; • recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, and everyday life.

Measurement Standard for Grades 6-8

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 6–8 all students should—
Understand measurable attributes of objects and the units, systems, and processes of measurement	<ul style="list-style-type: none">• understand both metric and customary systems of measurement;• understand relationships among units and convert from one unit to another within the same system;• understand, select, and use units of appropriate size and type to measure angles, perimeter, area, surface area, and volume.
Apply appropriate techniques, tools, and formulas to determine measurements	<ul style="list-style-type: none">• use common benchmarks to select appropriate methods for estimating measurements;• select and apply techniques and tools to accurately find length, area, volume, and angle measures to appropriate levels of precision;• develop and use formulas to determine the circumference of circles and the area of triangles, parallelograms, trapezoids, and circles and develop strategies to find the area of more-complex shapes;• develop strategies to determine the surface area and volume of selected prisms, pyramids, and cylinders;• solve problems involving scale factors, using ratio and proportion;• solve simple problems involving rates and derived measurements for such attributes as velocity and density.

Data Analysis and Probability Standard for Grades 6-8

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 6–8 all students should—
Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them	<ul style="list-style-type: none"> • formulate questions, design studies, and collect data about a characteristic shared by two populations or different characteristics within one population; • select, create, and use appropriate graphical representations of data, including histograms, box plots, and scatterplots.
Select and use appropriate statistical methods to analyze data	<ul style="list-style-type: none"> • find, use, and interpret measures of center and spread, including mean and interquartile range; • discuss and understand the correspondence between data sets and their graphical representations, especially histograms, stem-and-leaf plots, box plots, and scatterplots.
Develop and evaluate inferences and predictions that are based on data	<ul style="list-style-type: none"> • use observations about differences between two or more samples to make conjectures about the populations from which the samples were taken; • make conjectures about possible relationships between two characteristics of a sample on the basis of scatterplots of the data and approximate lines of fit; • use conjectures to formulate new questions and plan new studies to answer them.
Understand and apply basic concepts of probability	<ul style="list-style-type: none"> • understand and use appropriate terminology to describe complementary and mutually exclusive events; • use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations; • compute probabilities for simple compound events, using such methods as organized lists, tree diagrams, and area models.

Problem Solving Standard for Grades 6-8

Instructional programs from prekindergarten through grade 12 should enable all students to—

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard for Grades 6-8

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

Communication Standard for Grades 6-8

Instructional programs from prekindergarten through grade 12 should enable all students to—

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely.

Connections Standard for Grades 6-8

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics.

Representation Standard for Grades 6-8

Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Overview

Standards for Grades 6-8

The middle grades represent a significant turning point in students' lives. During the middle grades, students solidify conceptions about themselves as learners of mathematics. They arrive at conclusions about their competence in mathematics, their attitudes, their interest, and their motivation. These conceptions will influence how they approach the study of mathematics in later years, which in turn will affect their later career and personal opportunities.



If middle school students find both challenge and support in their mathematics classes, they will be drawn to the subject. They will be able to use their emerging capabilities of finding and imposing structure, conjecturing and verifying, thinking hypothetically, comprehending cause and effect, and engaging in abstraction and generalization.

As in all the grade bands, students in the middle grades need a balanced mathematics program that encompasses all ten Standards, including significant amounts of algebra and geometry. Algebra and geometry are crucial to success in the later study of mathematics and also in many situations that arise outside the mathematics classroom. Students should see that these subjects are interconnected with each other and with other content areas in the curriculum. For example, students might be asked to explain the number of tiles that will be needed to make borders around pools of various lengths and widths, as in figure 2. Students might develop various formulas to express this relationship on the basis of a table or their reasoning about the situation; for example, "You need $L + 2$ tiles across the top and the same number across the bottom. And you need W tiles on the left and the right. So all together, the number of tiles needed is $T = 2(L + 2) + 2W$."



Fig. 2. The "swimming pool" problem

Students' understanding of these crucial ideas should be developed over all three years in the middle grades and across a broad range of mathematics content. This approach is a challenging alternative to the practice of offering a select group of middle-grades students a one-year course that focuses narrowly on algebra or geometry. However, all middle-grades students will benefit from a rich and integrated treatment of mathematics content.

By the end of the eighth grade, students should have a solid background in algebra and other areas that will prepare them to enter substantive high school courses.

Middle-grades mathematics also needs to prepare students to deal with quantitative situations in their lives outside school. For example, consumer magazines regularly publish comparisons of characteristics of various consumer products, such as the quality of peanut butter, the duration of rechargeable batteries, or the cost, size, and gas mileage of automobiles. When using data from such sources, students need to determine which data are appropriate for their needs, to understand how the data were gathered at the source, and to consider limitations that could affect interpretation.

Special attention must be given to the preparation and ongoing professional support of middle-grades teachers. They need a deep understanding of mathematical ideas, pedagogical practices, interdisciplinary teaching approaches, how students learn mathematics, and adolescent development. States and provinces need to give much more attention to the development of special preparation programs for teachers of mathematics in the middle grades.

NCTM Standards (2000) for Grades 9th through 12th

Number and Operations Standard for Grades 9-12

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 9–12 all students should—
Understand numbers, ways of representing numbers, relationships among numbers, and number systems	<ul style="list-style-type: none">• develop a deeper understanding of very large and very small numbers and of various representations of them;• compare and contrast the properties of numbers and number systems, including the rational and real numbers, and understand complex numbers as solutions to quadratic equations that do not have real solutions;• understand vectors and matrices as systems that have some of the properties of the real-number system;• use number-theory arguments to justify relationships involving whole numbers.
Understand meanings of operations and how they relate to one another	<ul style="list-style-type: none">• judge the effects of such operations as multiplication, division, and computing powers and roots on the magnitudes of quantities;• develop an understanding of properties of, and representations for, the addition and multiplication of vectors and matrices;• develop an understanding of permutations and combinations as counting techniques.
Compute fluently and make reasonable estimates	<ul style="list-style-type: none">• develop fluency in operations with real numbers, vectors, and matrices, using mental computation or paper-and-pencil calculations for simple cases and technology for more-complicated cases.• judge the reasonableness of numerical computations and their results.

Algebra Standard for Grades 9-12

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 9–12 all students should—
Understand patterns, relations, and functions	<ul style="list-style-type: none"> • generalize patterns using explicitly defined and recursively defined functions; • understand relations and functions and select, convert flexibly among, and use various representations for them; • analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior; • understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more-complicated symbolic expressions; • understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions; • interpret representations of functions of two variables
Represent and analyze mathematical situations and structures using algebraic symbols	<ul style="list-style-type: none"> • understand the meaning of equivalent forms of expressions, equations, inequalities, and relations; • write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency—mentally or with paper and pencil in simple cases and using technology in all cases; • use symbolic algebra to represent and explain mathematical relationships; • use a variety of symbolic representations, including recursive and parametric equations, for functions and relations; • judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
Use mathematical models to represent and understand quantitative relationships	<ul style="list-style-type: none"> • identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships; • use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts; • draw reasonable conclusions about a situation being modeled.
Analyze change in various contexts	<ul style="list-style-type: none"> • approximate and interpret rates of change from graphical and numerical data

Geometry Standard for Grades 9-12

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 9–12 all students should—
Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships	<ul style="list-style-type: none"> • analyze properties and determine attributes of two- and three-dimensional objects; • explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them; • establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others; • use trigonometric relationships to determine lengths and angle measures.
Specify locations and describe spatial relationships using coordinate geometry and other representational systems	<ul style="list-style-type: none"> • use Cartesian coordinates and other coordinate systems, such as navigational, polar, or spherical systems, to analyze geometric situations; • investigate conjectures and solve problems involving two- and three-dimensional objects represented with Cartesian coordinates.
Apply transformations and use symmetry to analyze mathematical situations	<ul style="list-style-type: none"> • understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices; • use various representations to help understand the effects of simple transformations and their compositions.
Use visualization, spatial reasoning, and geometric modeling to solve problems	<ul style="list-style-type: none"> • draw and construct representations of two- and three-dimensional geometric objects using a variety of tools; • visualize three-dimensional objects and spaces from different perspectives and analyze their cross sections; • use vertex-edge graphs to model and solve problems; • use geometric models to gain insights into, and answer questions in, other areas of mathematics; • use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture.

Measurement Standard for Grades 9-12

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 9–12 all students should—
Understand measurable attributes of objects and the units, systems, and processes of measurement	<ul style="list-style-type: none"> • make decisions about units and scales that are appropriate for problem situations involving measurement.
Apply appropriate techniques, tools, and formulas to determine measurements	<ul style="list-style-type: none"> • analyze precision, accuracy, and approximate error in measurement situations; • understand and use formulas for the area, surface area, and volume of geometric figures, including cones, spheres, and cylinders; • apply informal concepts of successive approximation, upper and lower bounds, and limit in measurement situations; • use unit analysis to check measurement computations.

Data Analysis and Probability Standard for Grades 9-12

Expectations

Instructional programs from prekindergarten through grade 12 should enable all students to—	In grades 9–12 all students should—
Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them	<ul style="list-style-type: none"> • understand the differences among various kinds of studies and which types of inferences can legitimately be drawn from each; • know the characteristics of well-designed studies, including the role of randomization in surveys and experiments; • understand the meaning of measurement data and categorical data, of univariate and bivariate data, and of the term variable; • understand histograms, parallel box plots, and scatterplots and use them to display data; • compute basic statistics and understand the distinction between a statistic and a parameter.

<p>Select and use appropriate statistical methods to analyze data</p>	<ul style="list-style-type: none"> • for univariate measurement data, be able to display the distribution, describe its shape, and select and calculate summary statistics; • for bivariate measurement data, be able to display a scatterplot, describe its shape, and determine regression coefficients, regression equations, and correlation coefficients using technological tools; • display and discuss bivariate data where at least one variable is categorical; • recognize how linear transformations of univariate data affect shape, center, and spread; • identify trends in bivariate data and find functions that model the data or transform the data so that they can be modeled.
<p>Develop and evaluate inferences and predictions that are based on data</p>	<ul style="list-style-type: none"> • use simulations to explore the variability of sample statistics from a known population and to construct sampling distributions; • understand how sample statistics reflect the values of population parameters and use sampling distributions as the basis for informal inference; • evaluate published reports that are based on data by examining the design of the study, the appropriateness of the data analysis, and the validity of conclusions; • understand how basic statistical techniques are used to monitor process characteristics in the workplace.
<p>Understand and apply basic concepts of probability</p>	<ul style="list-style-type: none"> • understand the concepts of sample space and probability distribution and construct sample spaces and distributions in simple cases; • use simulations to construct empirical probability distributions; • compute and interpret the expected value of random variables in simple cases; • understand the concepts of conditional probability and independent events; • understand how to compute the probability of a compound event.

Problem Solving Standard for Grades 9-12

Instructional programs from prekindergarten through grade 12 should enable all students to—

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard for Grades 9-12

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

Communication Standard for Grades 9-12

Instructional programs from prekindergarten through grade 12 should enable all students to—

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely.

Connections Standard for Grades 9-12

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics.

Representation Standard for Grades 9-12

Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Overview

In secondary school, all students should learn an ambitious common foundation of mathematical ideas and applications. This shared mathematical understanding is as important for students who will enter the workplace as it is for those who will pursue further study in mathematics and science. All students should study mathematics in each of the four years that they are enrolled in high school.



Because students' interests and aspirations may change during and after high school, their mathematics education should guarantee access to a broad spectrum of career and educational options. They should experience the interplay of algebra, geometry, statistics, probability, and discrete mathematics. They need to understand the fundamental mathematical concepts of function and relation, invariance, and transformation. They should be adept at visualizing, describing, and analyzing situations in mathematical terms. And they need to be able to justify and prove mathematically based ideas.

High school mathematics builds on the skills and understandings developed in the lower grades. For example, students should enter high school with extensive experience in modeling various patterns and relationships. High school students might explore the following problem:

A student strained her knee in an intramural volleyball game, and her doctor prescribed an anti-inflammatory drug to reduce the swelling. She is to take two 220-milligram tablets every 8 hours for 10 days. If her kidneys filtered 60% of this drug from her body every 8 hours, how much of the drug was in her system after 10 days? How much of the drug would have been in her system if she had continued to take the drug for a year?

(See next page...)

	A	B
1	440	
2	616	
3	792.4	
4	968.8	
5	1145.2	
6	1321.6	
7	1498	
8	1674.4	
9	1850.8	
10	2027.2	
11	2203.6	
12	2380	
13	2556.4	
14	2732.8	
15	2909.2	
16	3085.6	
17	3262	
18	3438.4	
19	3614.8	
20	3791.2	
21	3967.6	
22	4144	
23	4320.4	
24	4496.8	
25		

Fig. 3. A spreadsheet computation of the "drug dosage" problem

Students might represent the equation informally as $\text{NEXT} = 0.4(\text{NOW}) + 440$, start at 440. Entering this relationship in a spreadsheet (see fig. 3), they could note that an "equilibrium" value of about 733 1/3 milligrams is reached. This investigation might lead to explorations of finite sequences and series.

High school students can study mathematics that extends beyond the material expected of all students in at least three ways. One is to include in the curriculum material that extends the foundational material in depth or sophistication. Two other approaches make use of supplementary courses. In the first, students enroll in additional courses concurrent with those expected of all students. In the second, students complete a three-year version of the shared material and then take other mathematics courses. In both situations, students can choose from such courses as computer science, technical mathematics, statistics, and calculus. Each of these approaches has the essential property that all students learn the same foundation of mathematics but some, if they wish, can study additional mathematics.

The Standards for high school students are ambitious. The demands made on high school teachers in achieving the Standards will require extended and sustained professional development and a large degree of administrative support.